THERMOCAPILLARY CONVECTION IN A HORIZONTAL LAYER OF LIQUID

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An exact solution is found for the equations for free convection in a planar horizontal layer of liquid with a constant temperature gradient at the boundaries. Two cases of boundary conditions for the velocity are considered: 1) the liquid is bounded by two solid planes, 2) the upper surface of the liquid is free, and the surface tension is a function of temperature.

1. Motion of the present type occurs, for example, in the middle of a large rectangular cell with a flat base. If one vertical side is hot and the opposite one is cold, the liquid cannot be in equilibrium, and motion occurs no matter how small the temperature difference may be. Two mechanisms give rise to convection. First, the density of the liquid at the hot wall is less than that at the cold one (if thermal expansion is normal). Second, the surface tension is a function of temperature, so capillary forces cause the liquid to move along the surface to the cold wall, with a compensating flow in the reverse direction at the bottom. The thickness of the layer determines which of these is the principal mechanism. Levich [1] has previously discussed purely capillary convection.

The temperature gradient at the boundaries may be considered as constant, at least as an average over the length, while the motion may be taken as parallel to the base. This allows us to consider the convection in an infinite plane-parallel horizontal layer with a constant horizontal temperature gradient at the boundaries.

Let the thickness of the layer of liquid be d = 2h. The x axis is taken as vertical, while the z axis runs from the hot wall to the cold one, the origin lying at the center of the liquid. The motion is independent of the y coordinate, which lies perpendicular to the xz plane.

The boundary conditions for the temperature may be put in the following form:

$$T = -Az \quad \text{for } x = \pm h. \tag{1.1}$$

Here the temperature T is reckoned relative to the mean value, while A is the horizontal temperature gradient at the boundaries of the liquid.

The speed of the steady-state motion is sought in the form

$$v_x = 0, \ v_y = 0, \ v_z = v(x). \tag{1.2}$$

From (1.2) the equations of free convection become

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = g\beta T, \qquad \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu\frac{\partial^2 v}{\partial x^2} \tag{1.3}$$

$$v\frac{\partial T}{\partial z} = \chi\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right). \tag{1.4}$$

Here p is pressure, ρ is the mean density of the liquid, ν is the kinematic viscosity, β is the coefficient of thermal expansion, χ is the thermal diffusivity, and g is the acceleration due to gravity.

Liquid	d_*, cm
Water H_2O	2.72
Glycerol $C_3H_5(OH)_3$	1.02
Methanol CH_3OH	1.04
Butanol C_4H_9OH	1.13
Diethyl ether $C_2H_5OC_2H_5$	1.12
Mercury Hg	1.06

The equation of continuity is complied with identically. The condition for the flow to be closed is

$$\int_{-h}^{h} v(x)\,dx = 0. \tag{1.5}$$

We eliminate the pressure from (1.3) to get

$$\frac{\partial^3 v}{\partial x^3} = \frac{g\beta}{\nu}\frac{\partial T}{\partial z}. \tag{1.6}$$

The left side is dependent only on x, which means that the temperature is a linear function of z. To find ∂T/∂z we differentiate (1.4) with respect to z and integrate twice with respect to x, obtaining

$$\frac{\partial T}{\partial z} = \frac{\nu}{g\beta}(c_1 x + c_2). \tag{1.7}$$

Substitution of (1.7) into (1.6) and integration with respect to x gives

$$v = c_1\frac{x^4}{4!} + c_2\frac{x^3}{3!} + c_3\frac{x^2}{2!} + c_4 x + c_5. \tag{1.8}$$

Integration of (1.7) with respect to z gives the temperature distribution as

$$T = \frac{\nu}{g\beta}(c_1 x + c_2)z + f(x), \tag{1.9}$$

where $f(x)$ must be found from (1.4),

$$\chi\frac{d^2 f}{dx^2} = v\frac{\partial T}{\partial z} \tag{1.10}$$

with a known right side. All the constants c_i of (1.8) and (1.9) must be found from (1.5) and the boundary conditions.

From (1.1) and (1.9) we get

$$c_1 = 0, \quad c_2 = -Ag\beta/\nu; \qquad f(\pm h) = 0. \tag{1.11}$$

The condition of (1.1) causes the expression for ∂T/∂z of (1.7) to become an even function of x. This problem may also be solved for the case where the horizontal temperature gradients at the boundaries are different.

2. Consider the free convection when the boundaries are formed by solids. The condition for adhesion

of the liquid to the boundaries gives

$$v = 0 \quad \text{for } x = \pm h. \tag{2.1}$$

Conditions (1.5), (1.11), and (2.1) uniquely determine all the constants of integration. Simple operations give us the velocity and temperature as

$$v = \frac{\nu}{h}\frac{G}{6}(\xi - \xi^3) \qquad \left(G = \frac{Ag\beta h^4}{\nu^2}\right) \qquad \left(\xi = \frac{x}{h}\right),$$
$$T = Ah\left[\frac{GP}{360}(3\xi^5 - 10\xi^3 + 7\xi) - \frac{z}{h}\right] \qquad \left(P = \frac{\nu}{\chi}\right). \tag{2.2}$$

Here G is the Grasshof number and P the Prandtl number.

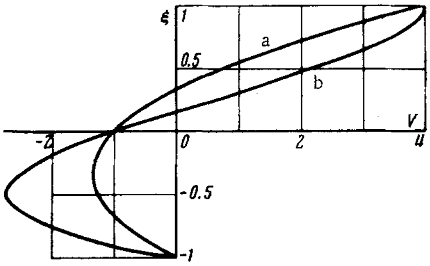

Fig. 1

Consider the heat flux in this layer of liquid. The flux Q per unit length along the y axis consists of two parts: the flux due to the thermal conductivity and the convective flux:

$$Q = -\varkappa \int_{-h}^{h} \frac{\partial T}{\partial z}\,dx + \rho c_p \int_{-h}^{h} vT\,dx = 2\varkappa Ah\left[1 + \frac{(GP)^2}{4725}\right] \tag{2.3}$$

in which $\varkappa$ and c_p are the thermal conductivity and specific heat, respectively.

There is also a vertical flux due to the motion, which may be deduced from the temperature gradient at the boundary. The flux through unit area of surface at x = h is

$$Q_1 = -\varkappa \left.\frac{\partial T}{\partial x}\right|_{x=h} = \frac{\varkappa AGP}{45}. \tag{2.4}$$

The same flux passes through unit area at x = −h.

An interesting point is that the velocity profile of (2.2) coincides with that for steady-state convection between vertical plates heated to different temperatures [2].

3. Consider the motion when the upper surface is free. The sum of the forces on unit area must here be zero. The frictional force, which is defined by the tensor for the viscous stresses, is accompanied by capillary forces due to the change in the surface tension. The condition for equilibrium may [1] be written as

$$\rho\nu\left(\frac{\partial v}{\partial x}\right)_{x=h} = \frac{\partial\sigma}{\partial T}\frac{\partial T}{\partial z}, \tag{3.1}$$

in which σ is the surface tension, which is taken to be a linear function of temperature.

As before, the velocity vanishes at the lower boundary,

$$v = 0 \quad \text{for } x = -h. \tag{3.2}$$

We use (3.1) and (3.2) with (1.5) to get from (1.8) and (1.9) that

$$v = \frac{\nu}{h}\left[\frac{G}{24}(-4\xi^3 + 3\xi^2 + 6\xi - 1) + \frac{G_\sigma}{24}(3\xi^2 + 2\xi - 1)\right]$$
$$T = Ah\left[\frac{GP}{480}(4\xi^5 - 5\xi^4 - 20\xi^3 + 10\xi^2 + 16\xi - 5) + \right.$$
$$\left. + \frac{G_\sigma P}{288}(-3\xi^4 - 4\xi^3 + 6\xi^2 + 4\xi - 3) - \frac{z}{h}\right]$$
$$\left(G_\sigma = \frac{3Ah^2}{\rho\nu^2}\left(-\frac{\partial\sigma}{\partial T}\right)\right). \tag{3.3}$$

Here the dimensionless parameter G_σ characterizes the motion due to the capillary action. The ratio

$$\frac{G_\sigma}{G} = \frac{3}{\rho g\beta h^2}\left(-\frac{\partial\sigma}{\partial T}\right) \tag{3.4}$$

indicates whether the density change or the capillary force dominates the convection. The first terms in (3.3) may be neglected if $G_\sigma/G \gg 1$, which corresponds to purely capillary convection; the velocity profile is then shown by curve (a) of Fig. 1. The first terms in (3.3) predominate if $G_\sigma/G \ll 1$, which corresponds to purely thermal convection, whose velocity profile is indicated by curve (b) of Fig. 1.

Figure 2 shows the velocity profile when the two mechanisms contribute equally. Curve (a) corresponds to normal thermal expansion, with $G_\sigma = G$; curve (b) corresponds to anomalous expansion ($\beta < 0$) and $G_\sigma = -G$. Here the velocity profile and the temperature distribution coincide with (2.2).

Relation (3.4) depends only on the parameter d = 2h for any given liquid. The characteristic thickness d* (that for which $G_\sigma/G = 1$) is given by

$$d_* = \left(-\frac{12}{\rho g\beta}\frac{\partial\sigma}{\partial T}\right)^{1/2}. \tag{3.5}$$

The table gives the d_* calculated for some liquids for a mean temperature of 25° C, the parameters being taken mainly from [3].

The convection is purely thermal for $d \gg d_*$, while it is largely capillary for $d \ll d_*$.

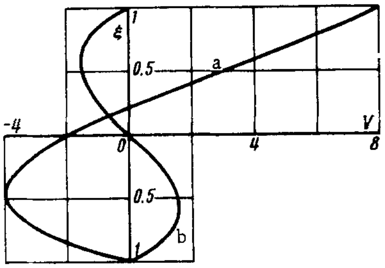

Fig. 2

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